MCA101 : COMPUTER GRAPHICS

2D GEOMETRY REPRESENTATION

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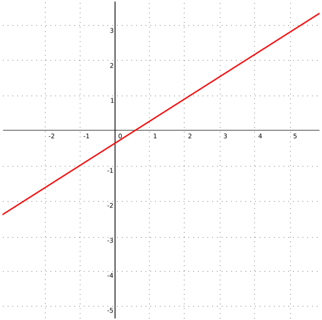
OUTLINE

OUTLINE

$y = mx + c$

$$y = f(x) = mx + c$$



PARAMETRIC FORM

For any two vectors $\mathbf{u}, \mathbf{v} \in V$, a point on the line segment joining them is given parameterised by $t \in [0, 1]$, as

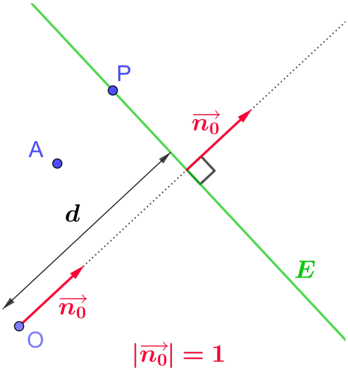
$$\mathbf{p} = f(t) = (1 - t)\mathbf{u} + t\mathbf{v}$$

PARAMETRIC FORM

Any point on a line in the direction of unit vector $\mathbf{u} : \|\mathbf{u}\|_2^2 = 1$, and an incident point $\mathbf{p}_0$ may be given parameterised by $t \in \mathbb{R}$ as,

$$\mathbf{p} = f(t) = \mathbf{p}_0 + t\mathbf{u}$$

HESSE NORMAL FORM



Distance from the origin O to the line E calculated with the Hesse normal form. Normal vector in red, line in green, point O shown in blue.

Given,
Normal to the line $\mathbf{n}_0 : \|\mathbf{n}_0\|_2^2 = 1$, and
its distance from origin, d;

The point on the line is given implicitly as the locus of all points $\mathbf{p}$ that satisfy,

$$\mathbf{n}_0 \cdot \mathbf{p} - d = 0$$

OUTLINE

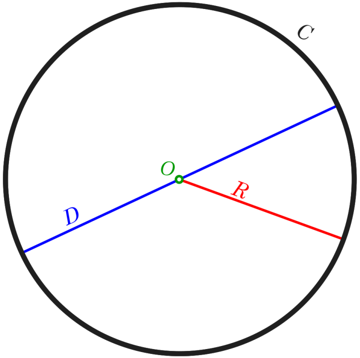


FIGURE: Image Courtesy: Wikipedia

Implicit Form:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2 - r^2 = 0$$

Parametric Form:

$$f(r, t) = \begin{bmatrix} r\cos t \\ r\sin t \end{bmatrix}$$

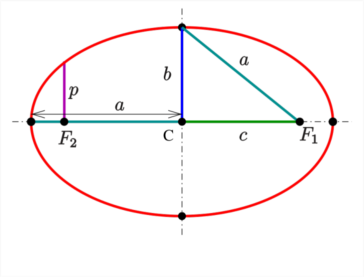


FIGURE: Image Courtesy: Wikipedia

Standard form

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

Parametric Form

$$f(t; a, b) = \begin{bmatrix} a\cos t \\ b\sin t \end{bmatrix}$$

OUTLINE

OUTLINE

PROBLEM

In a quantised (pixelated or discrete) 2d plane, find the set of points that visually approximate a given curve, say a straight line or a conic.

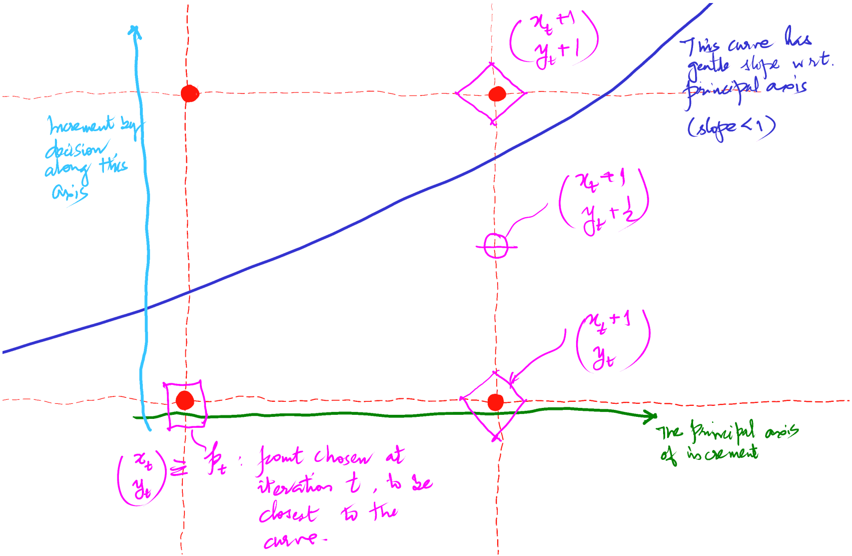
METHOD

Iteratively, increment along one axes, with respect to which, the slope of the curve is gentle.

Decide whether it is required to increment along the perpendicular axis or not.

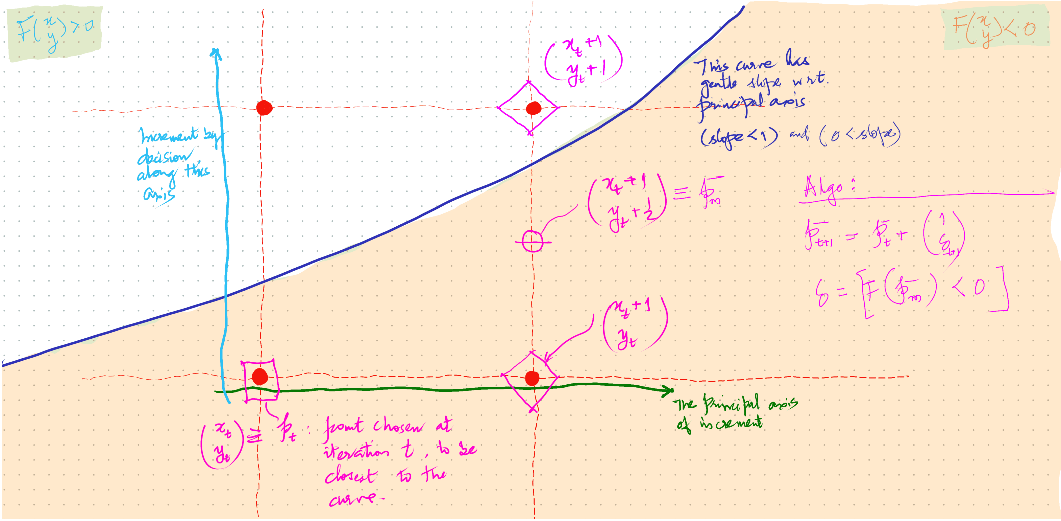
Increment if required.

EXAMPLE



$\begin{pmatrix} x_t+1 \\ y_t+1 \end{pmatrix}$
This curve has gentle slope w.r.t. principal axis
(slope < 1)
Increment by decision along this axis
$\begin{pmatrix} x_t+1 \\ y_t+\frac{1}{2} \end{pmatrix}$
$\begin{pmatrix} x_t+1 \\ y_t \end{pmatrix}$
The principal axis of increment
$\begin{pmatrix} x_t \\ y_t \end{pmatrix} \equiv p_t$: point chosen at iteration t, to be closest to the curve.

EXAMPLE



F(x, y) > 0
F(x, y) < 0
Increment by decision along this axis
(x_t+1, y_t+1)
This curve has gentle slope w.r.t. principal axis
(slope < 1) and (0 < slope)
(x_t+1, y_t+1/2) ≡ p_m
Algo:
p_{t+1} = p_t + (1, δ_{t+1})
δ = [F(p_m) < 0]
(x_t+1, y_t)
The principal axis of increment
(x_t, y_t) ≡ p_t : point chosen at iteration t, to be closest to the curve.

CONDITIONS FOR APPLICATION OF MID-POINT ALGORITHM

Mid-point algorithm is applicable to a curve within a given finite interval, iff

1. The curve increases monotonically;
2. The curve increases gradually.

In other words,

$$0 \leqslant \mathrm{d}y \leqslant \mathrm{d}x$$

GENERIC ALGO

Algorithm 1: Generic Mid-point Algorithm

Input: $x_0, x_{\max} \in \mathbb{Z}$ — Start and end x-coordinates.

Input: $F : \mathbb{R}^2 \to \mathbb{R}$ — Signed Distance Function from the curve.

Output: $C \equiv \{\mathbf{p}_0, \ldots, \mathbf{p}_{\max}\} \subset \mathbb{Z}^2$ — Curve in discrete 2D space.

$\mathbf{p}_0 \leftarrow \begin{bmatrix} x_0 \\ \lceil y_0 \rceil \end{bmatrix} \vdash F\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0$

for $t \in \{1, \ldots, \max\}$ **do**

- $\mathbf{p}_{\text{mid}} \leftarrow \mathbf{p}_{t-1} + \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$
- $\delta_t \leftarrow I[F(\mathbf{p}_{\text{mid}}) < 0]$
- $\mathbf{p}_t \leftarrow \mathbf{p}_{t-1} + \begin{pmatrix} 1 \\ \delta_t \end{pmatrix}$

OUTLINE

CHARACTERISING STRAIGHT LINES

$$F(x, y) = Ax - By + C$$

$$0 \leqslant \mathrm{d}y/\mathrm{d}x \leqslant 1 \quad \mapsto \quad 0 \leqslant A \leqslant B \qquad \text{...CASE 1}$$

$$-1 \leqslant \mathrm{d}y/\mathrm{d}x \leqslant 0 \quad \mapsto \quad 0 \leqslant A \leqslant -B \qquad \text{...CASE 2}$$

$$0 \leqslant \mathrm{d}x/\mathrm{d}y \leqslant 1 \quad \mapsto \quad 0 \leqslant B \leqslant A \qquad \text{...CASE 3}$$

$$-1 \leqslant \mathrm{d}x/\mathrm{d}y \leqslant 0 \quad \mapsto \quad 0 \leqslant -B \leqslant A \qquad \text{...CASE 4}$$

Read more [...]

MID-POINT ALGO FOR ST. LINE

Case 1. $0 < A < B$

Algorithm 2: Mid-Point Algorithm for Straight Line

Function MID-POINT-ALGO-ST-LINE-BASE (x_1, y_1, N, a, b, c) **is** — Base case.

- **Input:** $x_1, y_1, N \in \mathbb{Z} \vdash 0 < N$ — Start coordinates and num points.
- **Input:** $a, b, c \in \mathbb{Z} \vdash 0 \leqslant a < b$; b even — Coefficients: $F(x, y) = ax - by + c$.
- **Output:** $C \equiv \{(x_1, y_1), \ldots, (x_N, y_N)\} \subset \mathbb{Z}^2$

  An ordered sequence; a curve in discrete 2D space.

- $\delta_1 \leftarrow ax_1 - by_1 - \frac{b}{2} + c$ — $\frac{b}{2} \in \mathbb{Z}$ because b even.
- **for** $t \in \{2, \ldots, N\}$ **do**
  - $x_t \leftarrow x_{t-1} + 1$
  - $\delta_t \leftarrow \delta_{t-1} + a - b \cdot I[0 \leqslant \delta_{t-1}]$
  - $y_t \leftarrow y_{t-1} + I[0 \leqslant \delta_t]$
- **return** $C \equiv \{(x_1, y_1), \ldots, (x_N, y_N)\}$

MID-POINT ALGO FOR ST. LINE

Case 1. $0 < A < B$

Algorithm 3: Mid-Point Algorithm for Straight Line (alternate) [...CONTD]

Input: $x_1, y_1, N \in \mathbb{Z} \vdash 0 < N$ — Start and end x-coordinates.

Input: $a, b, c \in \mathbb{Z} \vdash 0 \leqslant a < b$; b even — Coefficients: $F(x, y) = ax - by + c$.

Output: $C \equiv \{(x_1, y_1), \ldots, (x_N, y_N)\} \subset \mathbb{Z}^2$

An ordered sequence; a curve in discrete 2D space.

```
Init: C ← ∅                              ▷ Initialise as array.
Init: (x, y, δ) ← (x1, y1, 0)            ▷ Initialise as integers.
δ ← ax − by − b/2 + c                    ▷ b/2 ∈ ℤ because b even.
C · PUSH((x, y))                         ▷ (x, y) is a tuple.
```

- **Init:** $C \leftarrow \emptyset$ — Initialise as array.
- **Init:** $(x, y, \delta) \leftarrow (x_1, y_1, 0)$ — Initialise as integers.
- $\delta \leftarrow ax - by - \frac{b}{2} + c$ — $\frac{b}{2} \in \mathbb{Z}$ because b even.
- $C \cdot \text{PUSH}((x, y))$ — (x, y) is a tuple.

MID-POINT ALGO FOR ST. LINE

Case 1. $0 < A < B$

Algorithm: [CONTD ...] Mid-Point Algorithm for Straight Line (alternate)

```
for t ∈ {2, . . . , N} do
    x ← x + 1                                  Increment along x-axis.
    δ ← δ + a − b · I[0 ⩽ δ]                    Update decision param δ.
    y ← y + I[0 ⩽ δ]                            Update along y-axis based on decision param.
    C · PUSH ((x, y))                           (x, y) is a tuple.
return C
```

MID-POINT ALGO FOR ST. LINE

Handle all cases (Case 2)

Algorithm 5: Mid Point Algorithm for Straight Lines (all cases) [...CONTD]

Function MID-POINT-ALGO-ST-LINE (x_1, x_{max}, a, b, c) **is**

A wrapper around MID-POINT-ALGO-ST-LINE-BASE

Input: $x_1, x_{max} \in \mathbb{Z} \vdash x_1 < x_{max}$ — Start and end x-coordinates.

Input: $a, b, c \in \mathbb{Z} \vdash a, b$ even — SDF $F(x, y) \triangleq ax - by + c$.

Output: $\mathbb{C} \equiv \{(x_1, y_1), \ldots, (x_{max}, y_{max})\}$ — Collection of points on curve.

$(y_1, y_{max}) \leftarrow \left(\left\lceil \frac{ax_1+c}{b} \right\rceil, \left\lceil \frac{ax_{max}+c}{b} \right\rceil\right)$ — Compute start and end y-coordinates.

if $0 < a < -b$ **then** — Case 2. $F'(x, y) = F(x, -y)$

  $(x_1, y_1, N, a, b, c) \leftarrow (x_1, -y_1, x_{max} - x_1 + 1, a, -b, c)$ — Flip about x-axis.

  define: TRF $((x, y)) \mapsto (x, -y)$ — Define inverse transformation.

Handle all cases (Case 3, 4)

Algorithm: [CONTD...] Mid Point Algorithm for Straight Lines (all cases) [...CONTD]

else if $0 < b < a$ **then** — Case 3. $F'(x, y) = F(-y, -x)$

  $(x_1, y_1, N, a, b, c) \leftarrow (-y_{max}, -x_{max}, -y_1 + y_{max} + 1, b, a, c)$ — Transpose.

  define: TRF $((x, y)) \mapsto (-y, -x)$ — Define inverse transformation.

else if $0 < -b < a$ **then** — Case 4. $F'(x, y) = F(-y, x)$

  $(x_1, y_1, N, a, b, c) \leftarrow (y_{max}, -x_{max}, y_1 - y_{max} + 1, -b, a, c)$ — Both flip and transpose.

  define: TRF $((x, y)) \mapsto (-y, x)$ — Define inverse transformation.

MID-POINT ALGO FOR ST. LINE

Handle all cases (Case 1)

Algorithm: [CONTD...] Mid Point Algorithm for Straight Lines (all cases)

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else — *Case 1. $0 < a < b$*

  $N \leftarrow x_{\max} - x_1 + 1$

  define: $\textsc{trf}\,((x, y)) \mapsto (x, y)$ — Identity.

$C \leftarrow \textsc{mid-point-algo-st-line-base}(x_1, y_1, N, a, b, c)$

$C \leftarrow \textsc{map}(\textsc{trf}, C)$

return C

EXERCISE 1

Using Bresenham's/ Mid-point Algorithm, compute the points along the following lines, between

1. $(2, 0) \to (6, 2)$
2. $(0, 1) \to (6, 13)$
3. $(0, 1) \to (6, -2)$
4. $(0, 4) \to (6, -8)$

SOLUTION 1 STEP 1

For each part, tabulate a, b, c, x_1, $x_{\max}$, y_1, $y_{\max}$, a', b', c', x'_1, $x'_{\max}$, y'_1, $y'_{\max}$, N.

Part	x_1	y_1	$x_{\max}$	$y_{\max}$	a	b	c	a'	b'	c'	x'_1	y'_1	$x'_{\max}$	$y'_{\max}$	N
1	2	0	6	2	1	2	-2	1	2	-2	2	0	6	2	5
2	0	1	6	13	2	1	1	1	2	1	-13	-6	-1	0	13
3	0	1	6	-2	1	-2	-2	1	2	-2	0	-1	6	2	7
4	0	4	6	-8	2	-1	-4	1	2	-4	-8	-6	4	0	13

SOLUTION 1 STEP 2

Compute the table of x, y, δ, $I[0 \leqslant \delta]$ for each iteration from 1 to N. Subsequently compute TRF (x, y).
Showing here for part 4.

x'_1	y'_1	N	a'	b'	c'	δ_1
-8	-6	13	1	2	-4	-1

Iteration	1	2	3	4	5	6	7	8	9	10	11	12	13
$x'_t \leftarrow x'_{t-1} + 1$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4
$\delta_t \leftarrow \delta_{t-1} + a' - b' c_{t-1}$	-1	0	-1	0	-1	0	-1	0	-1	0	-1	0	-1
$c_t \leftarrow I[0 \leqslant \delta_t]$	0	1	0	1	0	1	0	1	0	1	0	1	0
$y'_t \leftarrow y'_{t-1} + c_t$	-6	-5	-5	-4	-4	-3	-3	-2	-2	-1	-1	0	0
$x \leftarrow -y'$	6	5	5	4	4	3	3	2	2	1	1	0	0
$y \leftarrow x'$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4

EXERCISE 2 (PRACTICAL)

1. Can the calculations be done using an online spreadsheet with formulae?
2. Can the complete algorithm be encoded on a spreadsheet?

SOLUTION 2

[Link to Spreadsheet]

OUTLINE

SYMMETRY

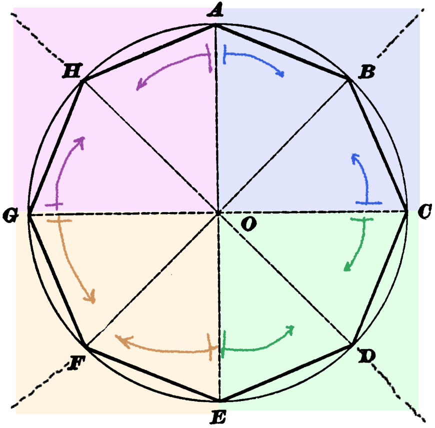


FIGURE: We choose 8-way symmetry of the circle for computational efficiency.

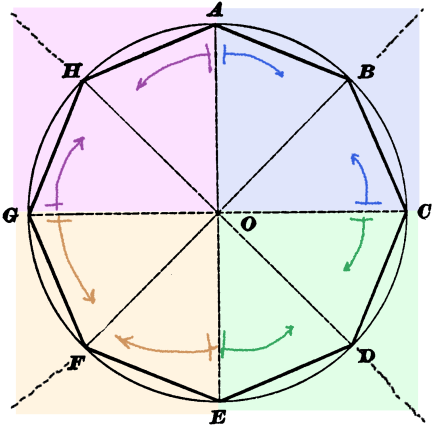
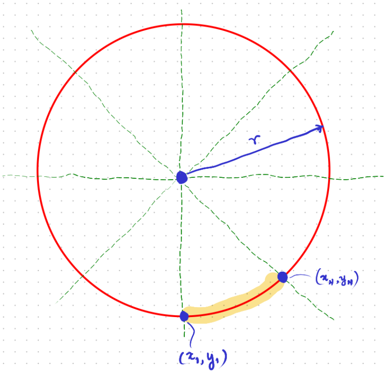


FIGURE: We choose 8-way symmetry of the circle for computational efficiency.

Computing the points on any one of the eight symmetrical sectors (octant), gives us the points on the other seven.
Let (x, y) be the point on one octant, the other seven are given as,

1. $(x, -y)$
2. $(-x, y)$
3. $(-x, -y)$
4. (y, x)
5. $(y, -x)$
6. $(-y, x)$
7. $(-y, -x)$

SETUP

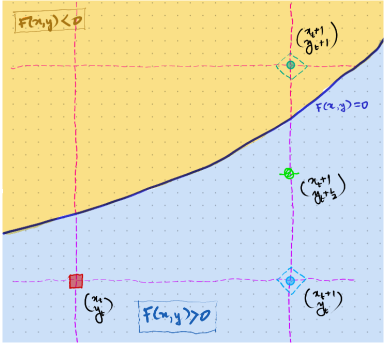


Given a circle with radius r, centred at origin, we intend to compute the points on (or nearest to) the circle within the octant defined by end points,

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -r \end{bmatrix}$$

$$\begin{bmatrix} x_N \\ y_N \end{bmatrix} = \begin{bmatrix} \left\lceil \frac{r}{\sqrt{2}} \right\rceil \\ -\left\lceil \frac{r}{\sqrt{2}} \right\rceil \end{bmatrix}$$

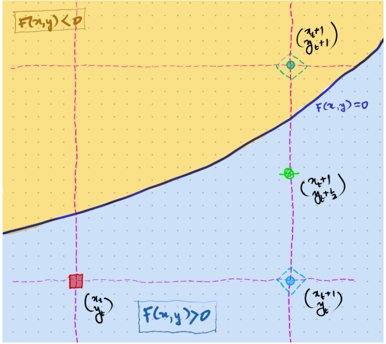
AT ITERATION t



$$F(x, y) = x^2 + y^2 - r^2$$

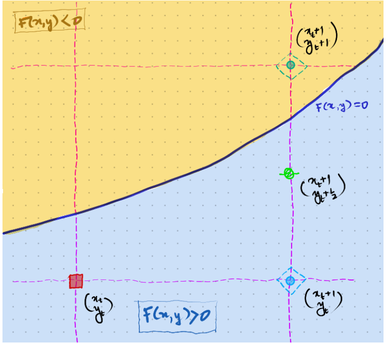
At timestep t, let (x_t, y_t) be the point closest to the curve given by $F(x, y) = 0$.

AT ITERATION $t+1$



$$x_{t+1} = x_t + 1$$
$$y_{t+1} = y_t + I[\delta_{t+1} > 0]$$

AT ITERATION $t+1$



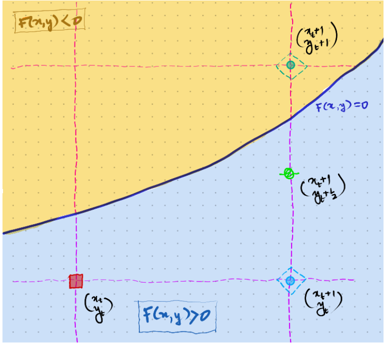
$$x_{t+1} = x_t + 1$$

$$y_{t+1} = y_t + I[\delta_{t+1} > 0]$$

$$\delta_{t+1} = F(x_t + 1, y_t + \frac{1}{2})$$

$$= x_t^2 + 2x_t + 1 + y_t^2 + y_t + \frac{1}{4} - r^2$$

AT ITERATION $t+1$



$$x_{t+1} = x_t + 1$$

$$y_{t+1} = y_t + I[\delta_{t+1} > 0]$$

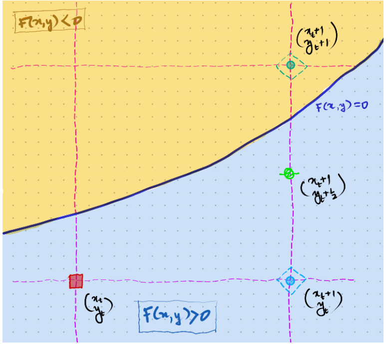
$$\delta_{t+1} = F(x_t + 1, y_t + \frac{1}{2})$$

$$= x_t^2 + 2x_t + 1 + y_t^2 + y_t + \frac{1}{4} - r^2$$

$$\delta_t = F(x_{t-1} + 1, y_{t-1} + \frac{1}{2}) = F(x_t, y_{t-1} + \frac{1}{2})$$

$$= x_t^2 + y_{t-1}^2 + y_{t-1} + \frac{1}{4} - r^2$$

AT ITERATION $t+1$



$$x_{t+1} = x_t + 1$$

$$y_{t+1} = y_t + I[\delta_{t+1} > 0]$$

$$\delta_{t+1} = F(x_t + 1, y_t + \frac{1}{2})$$

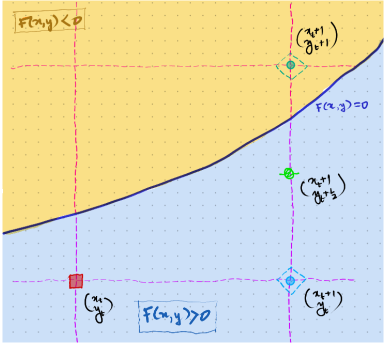
$$= x_t^2 + 2x_t + 1 + y_t^2 + y_t + \frac{1}{4} - r^2$$

$$\delta_t = F(x_{t-1} + 1, y_{t-1} + \frac{1}{2}) = F(x_t, y_{t-1} + \frac{1}{2})$$

$$= x_t^2 + y_{t-1}^2 + y_{t-1} + \frac{1}{4} - r^2$$

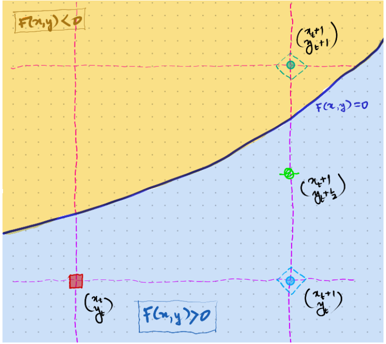
$$\delta_{t+1} - \delta_t = 2x_t + 1 + (y_t - y_{t-1})(y_t + y_{t-1} + 1)$$

AT ITERATION $t + 1$



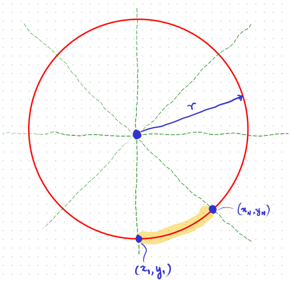
$$\begin{aligned} x_{t+1} &= x_t + 1 \\ y_{t+1} &= y_t + I[\delta_{t+1} > 0] \\ \delta_{t+1} - \delta_t &= 2x_t + 1 + (y_t - y_{t-1})(y_t + y_{t-1} + 1) \\ \delta_{t+1} &= \delta_t + 2x_t + 1 + I[\delta_t > 0] \cdot (2y_t) \end{aligned}$$

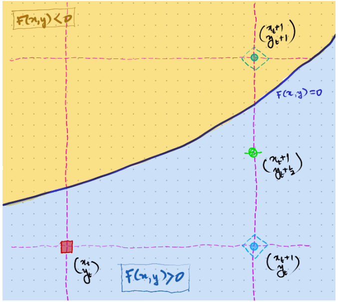
AT ITERATION $t+1$



$$
\begin{aligned}
{\color{red}x_{t+1}} &= x_t + 1 \\
{\color{red}y_{t+1}} &= y_t + I[\delta_{t+1} > 0] \\
{\color{red}\delta_{t+1}} &= \delta_t + 2x_t + 1 + I[\delta_t > 0] \cdot (2y_t)
\end{aligned}
$$

BOUNDARY CONDITIONS





$$\begin{bmatrix} x_1 & x_N \\ y_1 & y_N \end{bmatrix} = \begin{bmatrix} 0 & \left\lceil \frac{r}{\sqrt{2}} \right\rceil \\ -r & -\left\lceil \frac{r}{\sqrt{2}} \right\rceil \end{bmatrix}$$

$$\delta_2 = F(x_2, y_1 + \frac{1}{2}) \quad = F(1, -r + \frac{1}{2})$$
$$= 1 - r + \frac{1}{4} \quad \approx 1 - r$$

GENERIC ALGO

Algorithm 8: Generic Mid-point Algorithm

Input: $x_0, x_{\max} \in \mathbb{Z}$ — Start and end x-coordinates.

Input: $F : \mathbb{R}^2 \to \mathbb{R}$ — Signed Distance Function from the curve.

Output: $C \equiv \{\mathbf{p}_0, \ldots, \mathbf{p}_{\max}\} \subset \mathbb{Z}^2$ — Curve in discrete 2D space.

$$\mathbf{p}_0 \leftarrow \begin{bmatrix} x_0 \\ \lceil y_0 \rceil \end{bmatrix} \vdash F\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0$$

for $t \in \{1, \ldots, \max\}$ **do**

$$\mathbf{p}_{\text{mid}} \leftarrow \mathbf{p}_{t-1} + \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$$

$$\delta_t \leftarrow I[F(\mathbf{p}_{\text{mid}}) < 0]$$

$$\mathbf{p}_t \leftarrow \mathbf{p}_{t-1} + \begin{pmatrix} 1 \\ \delta_t \end{pmatrix}$$

MIDPOINT ALGO FOR CIRCLE OCTANT

Algorithm 9: Mid-Point Algorithm for Circle Octant

Function MID-POINT-ALGO-OCTANT (r) **is** — Base case.

  Input: $r \in \mathbb{Z} \vdash 0 < r$ — Radius of the circle.

  Output: $C \equiv \{(x_1, y_1), \ldots, (x_N, y_N)\} \subset \mathbb{Z}^2$

  An ordered sequence; a curve in discrete 2D space.

  $C \leftarrow \emptyset$ — Initialise array.

  $(x, y, \delta, N) \leftarrow (0, -r, 1 - r, \lceil \frac{r}{\sqrt{2}} \rceil)$ — Initialise with boundary condition.

  $C \cdot \text{PUSH}((x, y))$ — (x, y) is a tuple.

  for $x \in \{1, \ldots, N\}$ **do** — Iterate over x

    $y \leftarrow y + I[\delta > 0]$

    $C \cdot \text{PUSH}((x, y))$ — (x, y) is a tuple.

    $\delta \leftarrow \delta + 2x + 1 + I[\delta > 0] \cdot (2y)$

  return C

MIDPOINT ALGO FOR CIRCLE OCTANT

Algorithm 10: Mid-Point Algorithm for Circle

Function MID-POINT-ALGO-CIRCLE (r) **is** — All cases.

Input: $r \in \mathbb{Z} \vdash 0 < r$ — Radius of the circle.

Output: $C \equiv \{(x_1, y_1), \ldots, (x_N, y_N)\} \subset \mathbb{Z}^2$

An ordered sequence; a curve in discrete 2D space.

$C \leftarrow$ MID-POINT-ALGO-CIRCLE (r) — Initialise with octant points.

define: QUAD-SYM $((x, y)) \mapsto [(x, y), (x, -y), (-x, y), (-x, -y)]$

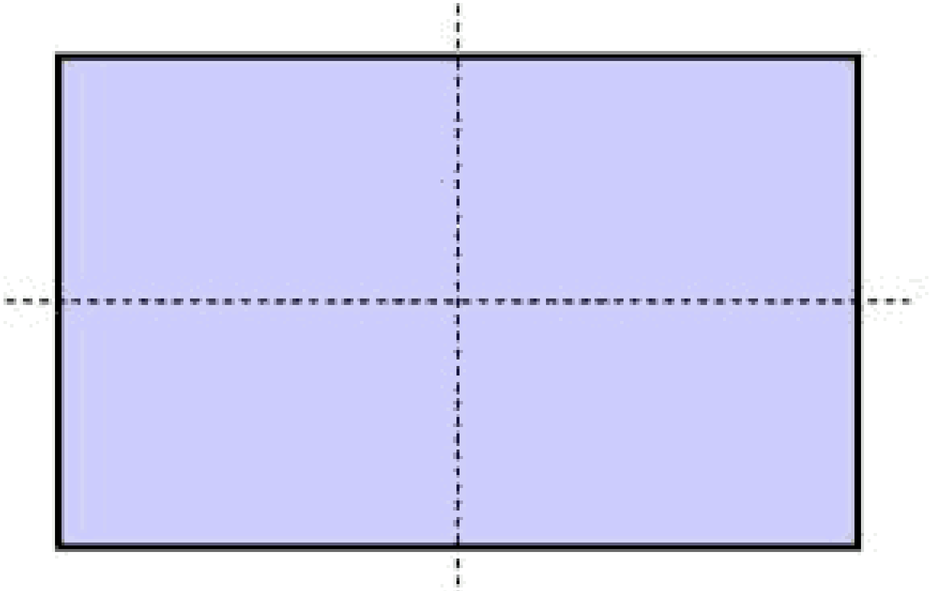
define: OCT-SYM $((x, y)) \mapsto$ CONCAT (QUAD-SYM $((x, y))$, QUAD-SYM $((y, x))$)

$C \leftarrow$ MAP-CONCAT (OCT-SYM, C)

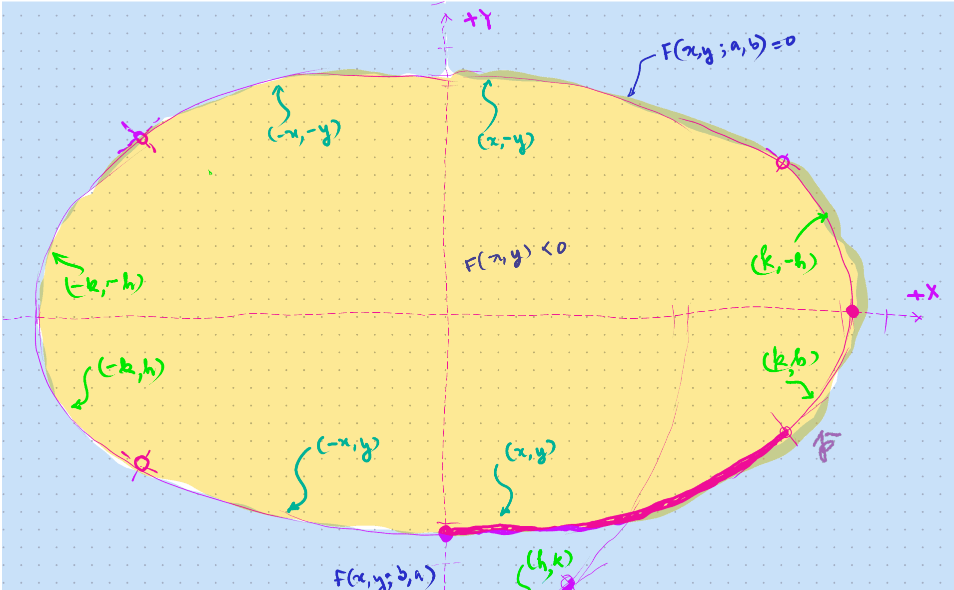
return C

OUTLINE

SYMMETRY



COMPUTATIONAL EFFICIENCY



SIGNED DISTANCE FUNCTION (SDF) OF ELLIPSE

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$F(x, y; a, b) = b^2 x^2 + z^2 y^2 - a^2 b^2$$

POINT OF INFLECTION

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{2x\mathrm{d}x}{a^2} + \frac{2y\mathrm{d}y}{b^2} = 0$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\mathbf{p}} = -\frac{b^2 x_p}{a^2 y_p} = 1$$

$$x_p = -\frac{a^2 y_p}{b^2}$$

$$= \pm\frac{a^2}{\sqrt{a^2 + b^2}}$$

AT ITERATION t

Let point (x_t, y_t) be closest to the theoretical curve,

AT ITERATION $t+1$

$$x_{t+1} = x_t + 1$$
$$y_{t+1} = y_t + I[\delta_{t+1} > 0]$$

AT ITERATION $t+1$

$$
\begin{aligned}
x_{t+1} &= x_t + 1 \\
y_{t+1} &= y_t + I[\delta_{t+1} > 0] \\
\delta_{t+1} &= F(x_t + 1, y_t + \frac{1}{2}) \\
&= b^2(x_t + 1)^2 + a^2(y_t + \frac{1}{2})^2 - a^2 b^2
\end{aligned}
$$

AT ITERATION $t+1$

$$
\begin{aligned}
x_{t+1} &= x_t + 1 \\
y_{t+1} &= y_t + I[\delta_{t+1} > 0] \\
\delta_{t+1} &= F(x_t + 1, y_t + \frac{1}{2}) \\
&= b^2(x_t + 1)^2 + a^2(y_t + \frac{1}{2})^2 - a^2 b^2 \\
&= b^2 x_t^2 + 2b^2 x_t + b^2 + a^2 y_t^2 + a^2 y_t + a^2 - a^2 b^2
\end{aligned}
$$

AT ITERATION $t+1$

$$
\begin{aligned}
x_{t+1} &= x_t + 1 \\
y_{t+1} &= y_t + I[\delta_{t+1} > 0] \\
\delta_{t+1} &= F(x_t + 1, y_t + \frac{1}{2}) \\
&= b^2(x_t + 1)^2 + a^2(y_t + \frac{1}{2})^2 - a^2b^2 \\
&= b^2x_t^2 + 2b^2x_t + b^2 + a^2y_t^2 + a^2y_t + a^2 - a^2b^2 \\
\delta_t &= b^2x_t^2 + a^2y_{t-1}^2 + a^2y_{t-1} + a^2 - a^2b^2
\end{aligned}
$$

AT ITERATION $t+1$

$$
\begin{aligned}
x_{t+1} &= x_t + 1 \\
y_{t+1} &= y_t + I[\delta_{t+1} > 0] \\
\delta_{t+1} &= F(x_t + 1, y_t + \frac{1}{2}) \\
&= b^2(x_t+1)^2 + a^2(y_t + \frac{1}{2})^2 - a^2b^2 \\
&= b^2x_t^2 + 2b^2x_t + b^2 + a^2y_t^2 + a^2y_t + a^2 - a^2b^2 \\
\delta_t &= b^2x_t^2 + a^2y_{t-1}^2 + a^2y_{t-1} + a^2 - a^2b^2 \\
\delta_{t+1} - \delta_t &= 2b^2x_t + b^2 + a^2(y_{t+1} - y_t)(y_{t+1} + y_t + 1)
\end{aligned}
$$

AT ITERATION $t+1$

$$\begin{aligned}
x_{t+1} &= x_t + 1 \\
y_{t+1} &= y_t + I[\delta_{t+1} > 0] \\
\delta_{t+1} - \delta_t &= 2b^2 x_t + b^2 + a^2 (y_{t+1} - y_t)(y_{t+1} + y_t + 1) \\
\delta_{t+1} &= \delta_t + 2b^2 x_t + b^2 + I[\delta_t > 0] \cdot (2a^2 y_t)
\end{aligned}$$

BOUNDARY CONDITIONS

at t=1, $x = 0$, $y = -b$
at t=2,

$$\begin{aligned}\delta_2 &= F(x_1 + 1, y_1 + \frac{1}{2}) \\ &= b^2 + a^2(-b + \frac{1}{2})^2 - a^2 b^2 \\ &= b^2 - a^2 b + \frac{a^2}{4}\end{aligned}$$

PUTTING IT ALL TOGETHER

$$
\begin{aligned}
x_1 &= 0 \\
y_1 &= -b \\
\delta_2 &= b^2 - a^2 b + \frac{a^2}{4} \\
N &= \frac{a^2}{\sqrt{a^2 + b^2}} \\
x_{t+1} &= x_t + 1 \\
y_{t+1} &= y_t + I[\delta_{t+1} > 0] \\
\delta_{t+1} &= \delta_t + 2b^2 x_t + b^2 + I[\delta_t > 0] \cdot (2a^2 y_t)
\end{aligned}
$$

MIDPOINT ALGORITHM FOR ELLIPSE

Algorithm 11: Mid Point Algorithm for Ellispe (BottomRight)

Function MID-POINT-ALGO-ELLIPSE-BR (a, b) **is** — Base Case.

  Input: $a, b \in \mathbb{Z} \vdash 0 < a, b$ — Semi-axes-lengths of the ellipse.

  Output: $C \equiv \{(x_1, y_1), \ldots, (x_N, y_N)\} \subset \mathbb{Z}^2$

  An ordered sequence; a curve in discrete 2D space.

  $C \leftarrow \emptyset$

  $(x, y, \delta) \leftarrow (0, -b, b^2 - a^2 b + \lceil \frac{a^2}{4} \rceil)$

  $C \cdot$ PUSH$((x, y))$

  $N \leftarrow \lceil \frac{a^2}{\sqrt{a^2+b^2}} \rceil$

  for $x \in \{1, \ldots, N\}$ **do** — Iterate along the x-axis.

    $y \leftarrow y + I[\delta > 0]$

    $C \cdot$ PUSH$((x, y))$

    $\delta \leftarrow \delta + 2b^2 x + b^2 + I[\delta > 0] \cdot (2a^2 y)$

  return C.

MIDPOINT ALGORITHM FOR ELLIPSE

Algorithm 12: Mid Point Algorithm for Ellipse (Collect Symmetric Points)

Function MID-POINT-ALGO-ELLIPSE (a, b) **is** — All cases.

Input: $a, b \in \mathbb{Z} \vdash 0 < a, b$ — Semi-axes-lengths of the ellipse.

Output: $C \equiv \{(x_1, y_1), \ldots, (x_N, y_N)\} \subset \mathbb{Z}^2$

An ordered sequence; a curve in discrete 2D space.

$C_1 \leftarrow$ MID-POINT-ALGO-ELLIPSE-BR (a, b) — Collect Bottom Right For (a,b)

define: QUAD-SYM : $((x, y)) \mapsto [(x, y), (x, -y), (-x, y), (-x, -y)]$

$C_1 \leftarrow$ MAP-CONCAT(QUAD-SYM, C_1) — Collect QUAD-SYM for C_1.

$C_2 \leftarrow$ MID-POINT-ALGO-ELLIPSE-BR (b, a) — Collect Bottom Right For (b,a)

define: QUAD-SYM-FLIP : $((x, y)) \mapsto$ QUAD-SYM$((y, x))$

$C_2 \leftarrow$ MAP-CONCAT(QUAD-SYM-FLIP, C_2) — Collect flipped QUAD-SYM for C_2.

$C \leftarrow$ CONCAT(C_1, C_2)

return C.